LAPLACE TRANSFORM

Definition: Let $f(t)$ be function defined for all positive values of t, then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Provided the integral exists, is called Laplace Transform of $f(t)$.

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \cdot dt}$$

Sufficient Condition - For Existence of Laplace Transform.

1)- $f(t)$ should be continuous, or piecewise continuous on the sub-interval of $(0, \infty)$

2) $f(t)$ should be exponential order.

i.e. $|f(t)| \leq M e^{\alpha t}$

where, $\alpha > 0$ is known as exponential Order

i.e $\lim_{t \to \infty} \left[F(t)\, e^{-\alpha t} \right] \xrightarrow{\text{finite}}$ for $t > \alpha$

and $f(t)$ is Continuous then Laplace Transform of $f(t)$ $\int_0^\infty e^{-st} f(t)\, dt$ exist.

Note: Above Condition are sufficient not necessary.

Important Example:.

① $L\{1\} = \frac{1}{s}$

② $L\{e^{at}\} = \frac{1}{s-a} : (s>a)$

③ $L\{t^n\} = \frac{n!}{s^{n+1}} \quad n \in \mathbb{N} \cup \{0\}$

④ $L\{\sin at\} = \frac{a}{s^2+a^2} ; (s>0)$

⑤ $L\{\cos at\} = \frac{s}{s^2+a^2} ; (s>0)$

⑥ $L\{\sinh at\} = \frac{a}{s^2+a^2} \quad (s>0)$

⑦ $L\{\cosh at\} = \frac{s}{s^2-a^2} \quad (s>0)$

Proof of Some.

Sol^n:.

$$L\{t^n\} = \int_0^\infty e^{-st} . f(t)\, dt$$

or
$$= \int_0^\infty e^{-st} . t^n\, dt$$

Now put
$st = x.$
$s\,dt = dx$
$t = 0 \Rightarrow x = 0$
$t = \infty \Rightarrow x = \infty$

or
$$= \int_0^\infty e^{-x} . \left(\frac{x}{s}\right)^n . \frac{dx}{s}$$

or
$$= \int_0^\infty \frac{e^{-x} . x^n}{s^n} . \frac{dx}{s}$$

or
$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$$

Gamma function -
$$\because \Gamma n = \int_0^\infty e^{-x} . x^{n-1}$$

or,
$$= \frac{1}{s^{n+1}} . \Gamma(n+1)$$

and, $\Gamma(n+1) = n!$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$